Autoregressive self-tuning feedback control of the Hénon map

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This paper describes a method based on self-feedback for controlling both chaotic and nonchaotic forms of the Hénon map with and without additive Gaussian white noise. We describe a nonlinear self-tuning controller that makes use of a feedback reference signal and a linear autoregressive formulation for the gain. This controller is effective at stabilizing the map to a variety of fixed point or period-two orbits. We contrast our approach with the method of Ott, Grebogi, and Yorke [Phys. Rev. Lett. **64**, 1196 (1990)] which has been used to control some chaotic processes and recently, some nonchaotic, stochastic ones. [S1063-651X(96)10912-0]

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I. INTRODUCTION

The chaos control method pioneered by Ott, Grebogi, and Yorke known as OGY [1] has proved effective in stabilizing unstable periodic orbits in several purportedly chaotic systems such as the rabbit septum and rat hippocampal slice preparation [2] using small parameter perturbations. Non-OGY strategies for taming chaos have also been developed as well (e.g., see [3]). In fact, since the introduction of OGY, there has been a huge increase in the number of published studies which utilize non-OGY as well as OGY methods to tame chaos (see [4] for an extensive bibliography of over 300 such references). Recently, a study by Christini and Collins [5] demonstrated that the OGY method could be used to identify and stabilize unstable fixed points (UFP) of a nonchaotic, stochastic system. The model they used was the second-order Hénon map with additive Gaussian white noise given by

$$x_{i+1}=1.0-Ax_i^2+Bx_{i-1}+\xi_i, \tag{1}$$

where A and B are system parameters, and ξ_i is independent $N(0,\sigma_\xi^2)$. For $B=0.3$, $A=1.0$, and $\xi_i=0$ this nonlinear map exhibits a stable period-four sequence $\{1.274\,98, -0.656\,35, 0.951\,69, -0.102\,63\}$. Fixed points of this map may be computed analytically from the quadratic equation

$$\overline{x^2}+0.7\overline{x}-1=0\ ,$$

and are

$$\overline{x_{1,2}}=\{0.7095,\ \ -1.4095\},$$

or may be determined empirically by identifying stable and unstable manifolds using the OGY method, for example.

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In this paper we describe an alternative control method and demonstrate how a reference feedback signal can be used to stabilize map (1) using an approach based on self-tuning control. This method has been used by us to successfully control a mathematical model of modulated cardiac parasystole [6] by applying linear, self-tuning, and nonlinear feedback controls to the system [7], and to control a quadratic map model of cardiac chaos [8]. Here, we use a linear autoregressive (AR)-type gain in a nonlinear self-tuning controller to stabilize nonchaotic as well as chaotic versions of the Hénon map with and without additive Gaussian white noise. One motivation for this work is that certain physiological systems produce stochastic, nonchaotic (nonlinear) behavior that, nevertheless can be stabilized using a chaos control technique such as OGY (as demonstrated in [5]) or by using the approach described herein.

II. METHOD

The controlled form of map (1) is

$$x_{i+1}=1.0-Ax_i^2+Bx_{i-1}+\xi_i+g_ie_i, \tag{2}$$

where g_i is a self-tuning control gain to be automatically determined at each step, and $e_i=x_i-v_{ref}$ is the tracking error with respect to a constant amplitude reference signal v_{ref}. Figure 1 is a schematic diagram of the controlled map (2). A traditional self-tuning controller is one in which the control gain is modified by the feedback error on each iteration. The objective of the design is to find a simple and implementable g_i to achieve the goal of automatic control, i.e.,

$$x_i\rightarrow\overline{x}\quad\text{as}\quad i\rightarrow\infty,$$

where $\overline{x}$ is the desired target state which is usually (but not necessarily) a UFP, $\overline{x_{\rm UFP}}$, of the original map [9], perhaps of

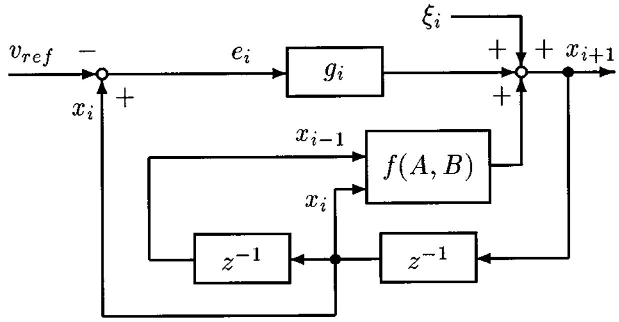

FIG. 1. Schematic diagram of the controlled Hénon map. The z^{-1} blocks each represent a one iteration delay unit.

large period. When the control objective is finally realized, $x_i-\overline{x}$ will be equal to a small (ideally zero) constant at the time iteration halts. We desire a g_i that does not depend explicitly on *a priori* system knowledge or prediction of future system behavior. Rather, we prefer a g_i that depends only on past system performance.

A simple design for the self-tuning gain is one that uses a linear combination of the supplied reference signal and a few available previous system states. It satisfies the following linear AR-type relation:

$$g_i=kv_{ref}+\sum_{j=0}^{n} a_j x_{i-j}\,, \tag{3}$$

where n is a small integer, independent of the system dimension, and the k and a_j's are to be determined for stable tracking to the target $\overline{x}$.

A mathematical justification is as follows. With $\xi_i=0$ and g_i given by Eq. (3), we rewrite Eq. (2) as

$$x_{i+1}=1.0-Ax_i^2+Bx_{i-1}+\left(kv_{ref}+\sum_{j=0}^{n} a_j x_{i-j}\right)(x_i-v_{ref}). \tag{4}$$

Notice that the controller itself (g_ie_i) is nonlinear. We consider the special case $n=1$ which will demonstrate clearly the general idea of our approach. Equation (4) reduces to

$$\begin{aligned} x_{i+1}&=(1.0-kv_{ref}^2)+(a_0-A)x_i^2+a_1x_{i-1}x_i\\ &\quad+(kv_{ref}-a_0v_{ref})x_i+(B-a_1v_{ref})x_{i-1}\\ &=\alpha+\beta x_i^2+\gamma x_ix_{i-1}+\mu x_i+\eta x_{i-1}, \end{aligned} \tag{5}$$

where

$$\alpha=1.0-kv_{ref}^2,\quad \beta=a_0-A,\quad \gamma=a_1,$$

$$\mu=kv_{ref}-a_0v_{ref},\quad \eta=B-a_1v_{ref}.$$

If we wish to control the orbit x_i specifically to the target $\overline{x}_{\rm UFP}$ of map (1) (it is known that a stable fixed point is fairly easy to arrive at), it may prove to be very difficult to use $\overline{x}_{\rm UFP}$ itself as a term in the feedback controller. This is because a UFP is highly sensitive to noise and small perturbations, and thus difficult to realize physically. Instead we make use of the reference signal v_{ref} to achieve the goal $x_i\to\overline{x}$ as $i\to\infty$.

Let's assume that the control objective has been successfully achieved. It then follows from Eq. (5) that

$$\overline{x}=\alpha+\beta\overline{x}^2+\gamma\overline{x}^2+\mu\overline{x}+\eta\overline{x}$$

or

$$(\beta+\gamma)\overline{x}^2+(\mu+\eta-1)\overline{x}+\alpha=0\,. \tag{6}$$

This indicates that the control target $\overline{x}$, originally a UFP of Eq. (1), is now a stable fixed point of Eq. (6) whose solutions are given by

$$\overline{x}=\frac{(1-\mu-\eta)\pm\sqrt{(1-\mu-\eta)^2-4\alpha(\beta+\gamma)}}{2(\beta+\gamma)}. \tag{7}$$

Necessary conditions for the existence of two real, finite, fixed points of Eq. (6) are

$$2(\beta+\gamma)\neq 0 \quad \text{or} \quad (a_0+a_1)\neq A, \tag{8}$$

and

$$(1-\mu-\eta)^2-4\alpha(\beta+\gamma)>0,$$

or

$$[v_{ref}(k-a_0-a_1)-1+B]^2-4(1-kv_{ref}^2)(a_0+a_1-A)>0 \tag{9}$$

This also provides bounds on v_{ref} as a function of k, a_0, a_1 and the two system parameters to ensure stable feedback control. For example, if we set $a_0=a_1=0.4$ and $k=0.8$, then Eq. (9) reduces to the inequality $|v_{ref}|<1.4197$.

If we now insist that $\overline{x}=\overline{x}_{\rm UFP}$ as $i\to\infty$ then we must simultaneously satisfy

$$A\overline{x}_{\rm UFP}^2+(1-B)\overline{x}_{\rm UFP}-1.0=0 \tag{10}$$

and

$$(\beta+\gamma)\overline{x}_{\rm UFP}^2+(\mu+\eta-1)\overline{x}_{\rm UFP}+\alpha=0, \tag{11}$$

in which $\overline{x}_{\rm UFP}$ is an unstable solution of Eq. (10) but a stable solution of Eq. (11). It is well known that for the Hénon map [Eq. (10)],

$$\overline{x}_{\rm UFP}=\frac{(B-1)+\sqrt{(B-1)^2+4A}}{2A}. \tag{12}$$

Hence, to control the system trajectory to this target, we substitute it into Eq. (11) and obtain

$$\begin{aligned} &\alpha+(\beta+\gamma)\left(\frac{(B-1)+\sqrt{(B-1)^2+4A}}{2A}\right)^2+(\mu+\eta-1)\\ &\quad\times\left(\frac{(B-1)+\sqrt{(B-1)^2+4A}}{2A}\right)=0\,. \end{aligned} \tag{13}$$

This indicates simply that v_{ref} must be chosen to be $\overline{x_{\mathrm{UFP}}}$ in this case [substitute $\overline{x_{\mathrm{UFP}}}$ for v_{ref} in Eq. (13) to recover Eq. (10)].

To complete the analysis, we determine a sufficient condition for the control of x_i to the target $\overline{x_{\mathrm{UFP}}}$ by choosing the reference signal $v_{ref}=\overline{x_{\mathrm{UFP}}}$. Substituting $v_{ref}=\overline{x_{\mathrm{UFP}}}$ in Eq. (5) gives

$$x_{i+1}=1.0-Ax_i^2+Bx_{i-1}+(k\overline{x_{\mathrm{UFP}}}+a_0x_i+a_1x_{i-1})\times(x_i-\overline{x_{\mathrm{UFP}}}).$$

Note that $\overline{x_{\mathrm{UFP}}}$ satisfies the original system, i.e.,

$$\overline{x_{\mathrm{UFP}}}=1.0-A\overline{x_{\mathrm{UFP}}^2}+B\overline{x_{\mathrm{UFP}}}.$$

A subtraction of these last two equations gives

$$e_{i+1}=(a_0-A)e_i^2+a_1e_ie_{i-1}+(k+a_0+a_1-2A)(\overline{x_{\mathrm{UFP}}}e_i)+Be_{i-1},$$

where $e_{i+1}=x_{i+1}-\overline{x_{\mathrm{UFP}}}$. Let $\varepsilon_i=e_{i-1}$ and then rewrite the last equation as

$$e_{i+1}=(a_0-A)e_i^2+a_1e_i\varepsilon_i+(k+a_0+a_1-2A),\times(\overline{x_{\mathrm{UFP}}}e_i)+B\varepsilon_i, \tag{14}$$

$$\varepsilon_{i+1}=e_i.$$

Since $\overline{x_{\mathrm{UFP}}}$ is a constant, this equation is ''autonomous.'' Thus we make use of the Lyapunov first (or indirect) method to determine the stability of the system [10]. Following a similar analysis of the Hénon system in [11], we calculate its Jacobian evaluated at $e_i=\varepsilon_i=0$ as

$$J=\begin{bmatrix}(k+a_0+a_1-2A)\overline{x_{\mathrm{UFP}}} & B\\ 1 & 0\end{bmatrix},$$

whose eigenvalues are

$$\lambda_{1,2}=\frac{(k+a_0+a_1-2A)\overline{x_{\mathrm{UFP}}}\pm\sqrt{(k+a_0+a_1-2A)^2\overline{x_{\mathrm{UFP}}^2}+4B}}{2}.$$

As a result, a sufficient condition for control is

$$|\lambda_{1,2}|=\left|\frac{(k+a_0+a_1-2A)\overline{x_{\mathrm{UFP}}}\pm\sqrt{(k+a_0+a_1-2A)^2\overline{x_{\mathrm{UFP}}^2}+4B}}{2}\right|<1, \tag{15}$$

for a suitable choice of the parameters.

Finally, it should be pointed out that if we are willing to relax the requirement that we control to $\overline{x_{\mathrm{UFP}}}$, then we can make use of Eqs. (8) and (9) to specify one of a range of possible values for v_{ref} and thus tune the control outcome to one of a set of $\overline{x}$'s depending on the specific values chosen for the parameters v_{ref}, k, a_0, and a_1. Note that while Eqs. (8) and (9) do not require knowledge of $\overline{x_{\mathrm{UFP}}}$ they do require that we know the values of the system parameters (in this case A and B). However, we can make use of certain descriptive measures of system state to come up with reasonable choices for the values of the four parameters. We demonstrate this in Sec. III [12].

III. RESULTS

Here we present examples of period-one and period-two control under four different cases of map (1), namely, (1) nonchaotic, nonstochastic, (2) nonchaotic, stochastic, (3) chaotic, nonstochastic, and (4) chaotic, stochastic. The gain parameters are chosen to be the same for all cases using the following strategy (for $n=1$): we specify that $a_0=a_1=C$, $k=2C$, and $C<A/2$ (C constant). With these choices Eq. (9) simplifies to

$$|v_{ref}|<\frac{(B-1)^2+4A-8C}{8C(A-2C)}. \tag{16}$$

This gives the stable range of values for v_{ref} as a function of the system and gain parameters.

Case 1. nonchaotic, nonstochastic

Here we use Eq. (3) to attempt to control Eq. (1) with $\xi_i=0$. For the nonchaotic case of $A=1$ and $B=0.3$, then $C<0.5$. From Eq. (9) we have

$$|v_{ref}|<\frac{4.49-8C}{8C(1-2C)}.$$

Consider the case $C=0.4$ ($k=0.8$). Note that these values satisfy Eqs. (15). Then $|v_{ref}|<1.4197$ which encompasses the original system state range ($-0.656\,35<x_i<1.274\,98$). Thus stability will be assured if we select any v_{ref} in this range as well. Figure 2(a) is a bifurcation-style plot of $\overline{x}$ vs v_{ref} showing two regions where control is period-one, and a region ($-1.2<v_{ref}<0.5$) where control is period-two. The gain is

$$g_i=0.8v_{ref}+0.4x_i+0.4x_{i-1}. \tag{17}$$

Substituting Eq. (17) in Eq. (6) yields

$$0.2\overline{x^2}+0.7\overline{x}-(1.0-0.8v_{ref}^2)=0. \tag{18}$$

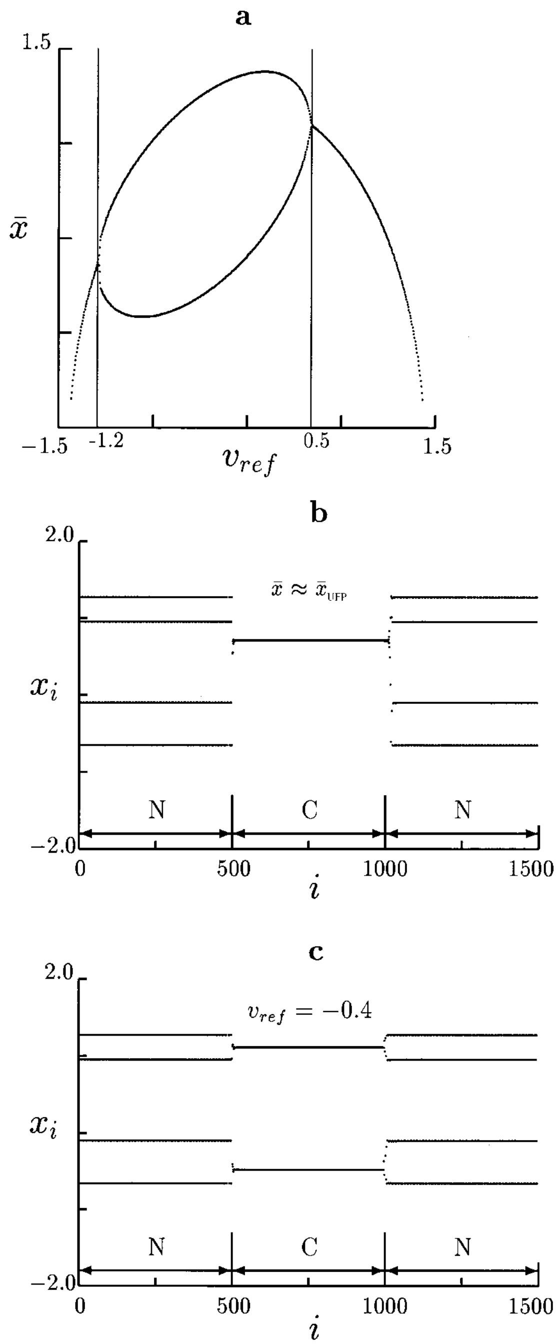

FIG. 2. Control of nonchaotic, nonstochastic case ($A=1$, $B=0.3$, $\xi_i=0$). (a) Bifurcation-type plot showing period-one and period-two control regions. (b) Period-one control for $v_{ref}=0.7095\approx\overline{x_{\rm UFP}}$. (c) Period-two control for $v_{ref}=-0.4$. Here, and in the remaining figures, *N* are noncontrol periods, *C* control period.

For example, for $v_{ref}=0.7095\approx\overline{x_{\rm UFP}}$ Eq. (18) simplifies to

$$0.2\overline{x^2}+0.7\overline{x}-0.5973=0,$$

which has a stable root at $\overline{x}=0.7095$ in accord with Eq. (15).

Figure 2(b) shows results of the controlled system Eq. (6) with $v_{ref}=0.7095$. When the control is initiated at $i=500$, stabilization occurs rapidly (within 27 iterations to 4 decimal places) to $x_i=0.7095$. When control is turned off at

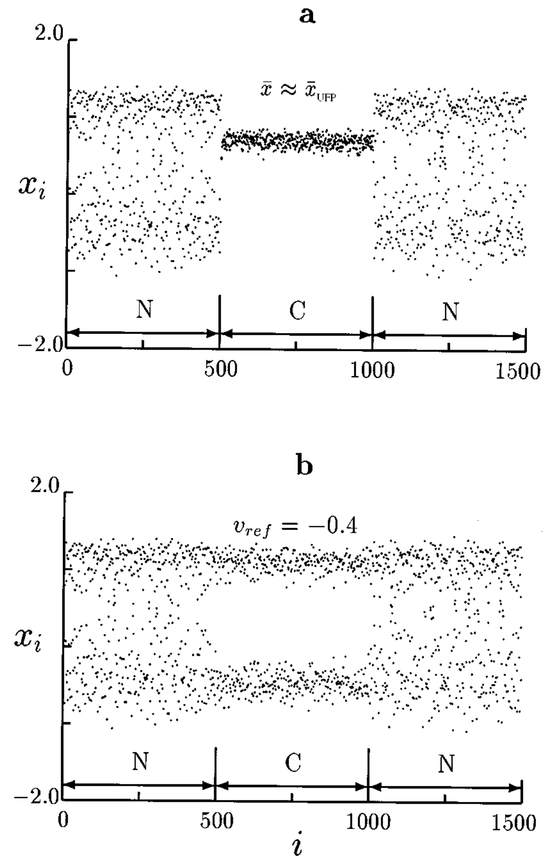

FIG. 3. Control of nonchaotic, stochastic case ($A=1$, $B=0.3$, $\sigma_\xi\approx0.04$). (a) Period-one control for $v_{ref}=0.7095\approx\overline{x_{\rm UFP}}$. (b) Period-two control for $v_{ref}=-0.4$.

$i=1000$ there is a rapid return to the original period-four output. Figure 2(c) shows results for $v_{ref}=-0.4$ which yields period-two control as shown in Fig. 2(a). There are two observations to be noted from these results: (1) A variety of period-one and period-two target trajectories can be reliably achieved in addition to control to $\overline{x_{\rm UFP}}$. (2) From Eq. (16) we see that the possible values for v_{ref} can be safely chosen to be within the data range of the observed system state, that is $-0.656\,35<x_i<1.274\,98$ which is well within $|v_{ref}|<1.4197$.

Case 2. nonchaotic, stochastic

In [5] Christini and Collins demonstrated that the OGY method can be used to control Eq. (1) for this particular case. We again specify $A=1$, $B=0.3$ and select ξ_i nonzero with $\sigma_\xi\approx0.04$ [13] in Eq. (2) and again $v_{ref}=0.7095\approx\overline{x_{\rm UFP}}$. The control results are as shown in Fig. 3(a). The noise at this level does not appear to interfere with the ability to control the system using self-tuning feedback. Figure 3(b) shows results of control of Eq. (5) again for $v_{ref}=-0.4$ to a period-two $\overline{x}$.

Case 3. chaotic, nonstochastic

For this case, we set $A=1.2$, $B=0.3$, and $\xi_i=0$ in Eq. (1). For the gain Eq. (3) we again set $a_0=a_1=0.4$, and $k=0.8$ as in cases 1 and 2. Substituting in Eq. (9) gives

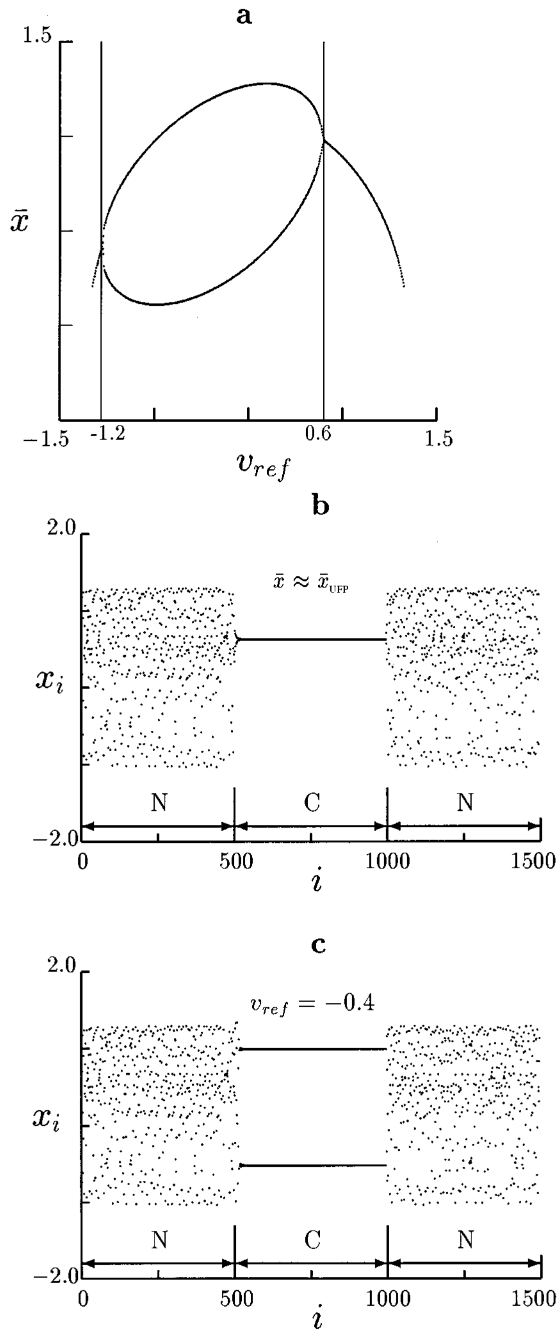

FIG. 4. Control of chaotic, nonstochastic case ($A=1.2$, $B=0.3$, $\xi_i=0$). (a) Bifurcation-type plot showing period-one and period-two control regions. (b) Period-one control for $v_{ref}=0.7095\approx\overline{x}_{\mathrm{UFP}}$. (c) Period-two control for $v_{ref}=-0.4$.

$$0.49+1.6(1.0-0.8v_{ref}^2)>0$$

yielding $|v_{ref}|<1.2489$. Figure 4(a) is the bifurcation-type diagram of $\overline{x}$ vs $(-1.24\leqslant v_{ref}\leqslant 1.24)$ for this case. If we select $v_{ref}=0.7095$ and solve $0.4\overline{x}^2+0.7\overline{x}-0.2=0$ the result is a stable root at $\overline{x}=0.7095$. Figure 4(b) shows the control of this chaotic map for $v_{ref}=0.7095$ which indeed leads to the final trajectory $\overline{x}=0.7095$. Figure 4(c) shows results of control of Eq. (5) for $v_{ref}=-0.4$ to a period-two $\overline{x}$ with the final $\overline{x}$ as indicated in Fig. 4(a).

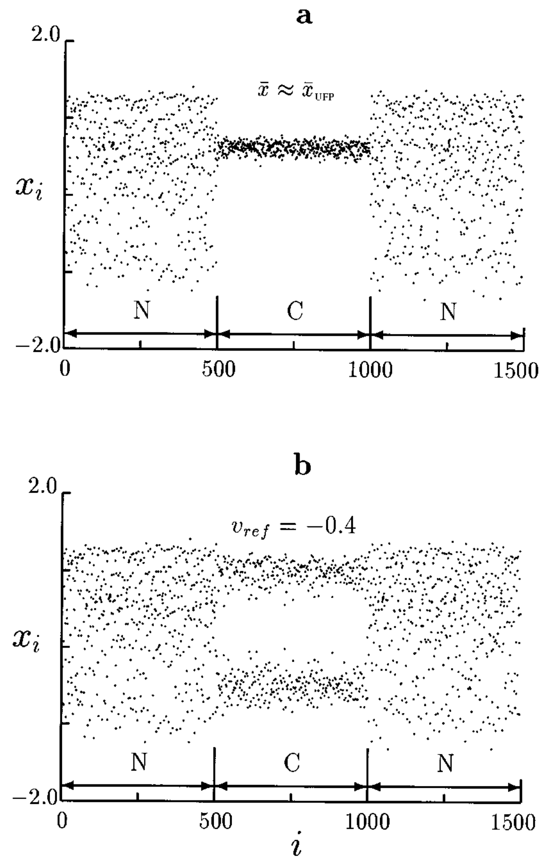

FIG. 5. Control of chaotic, stochastic case ($A=1.2$, $B=0.3$, $\sigma_\xi\approx 0.035$). (a) Period-one control for $v_{ref}=0.7095\approx\overline{x}_{\mathrm{UFP}}$. (b) Period-two control for $v_{ref}=-0.4$.

Case 4. chaotic, stochastic

Here we use the same parameters as in case 3 with the exception that ξ_i is nonzero and $\sigma_\xi\approx 0.035$ (σ_ξ greater than about 0.04 causes destabilization of the map). Figure 5(a) shows the result of control for $v_{ref}=0.7095$, and Fig. 5(b) shows results of control of Eq. (5) for $v_{ref}=-0.4$ to a period-two $\overline{x}$. Again, the additive noise at this level does not interfere with the ability to control the system.

IV. DISCUSSION AND CONCLUSIONS

We presented a method based on nonlinear feedback control with (linear) AR-type gain for stabilizing the Hénon map in chaotic and nonchaotic modes, with and without random noise. In [11], it was shown that a linear feedback controller was unable to stabilize the Hénon map in all four configurations studied in the present paper. Linear feedback controllers are generally not capable of stabilizing nonlinear systems (including chaotic ones) and are likely unable to control chaotic plus stochastic systems.

Christini and Collins [5] demonstrated that the OGY method could be used to control a nonchaotic, stochastic Hénon map to its unstable periodic orbits, as we also show with our approach. Pyragas and Tamaševičius [14] were among the first researchers to achieve success in using output feedback mainly to control nonlinear continuous time systems, however systematic procedures for determining the feedback gain have been problematical. There are several key features of our approach: (1) In addition to the UFP of the original system, a range of fixed point and period-two trajectories can be controlled using our method, which has been confirmed by our many simulations. This provides a wide latitude in choosing a desired target trajectory and therefore represents an advantage over the OGY method. (2) The method can be used to control any combination of chaotic, nonchaotic, stochastic and nonstochastic forms of the Hénon map. (3) Based on our various computer simulations, it is robust with respect to noise inputs. (4) In using an easily tuned reference signal v_{ref} in the design of the feedback controller (without necessarily having to use the physically sensitive unstable periodic orbit or fixed point value of the given system), we have improved both the design and implementation of the feedback control technique developed by Chen and Dong [15] for similar tasks. As a possible strategy for selecting v_{ref} from Eq. (9) we note that if we replace parameter B by the observed minimum of x_i (-0.656) and parameter A by the observed maximum (1.275), and select $C=0.4$, then $|v_{ref}|<3.054$ which again would encompass the data range. This suggests that we may be able to safely substitute a certain subrange of the output data amplitude for the allowable v_{ref}'s, for example ± 1 standard deviation of the mean of the data measured over a suitable precontrol interval, with $0<C<0.5$.

An alternative approach for selecting the gain parameters would be to apply AR (or other) modeling to the data over some stationary precontrol interval and then use the computed coefficients for the gain parameters themselves. These and related approaches will clearly require testing on arbitrary nonlinear and nonlinear-stochastic systems.

Finally, we point out that even though feedback control methods in general require some knowledge or estimation of the underlying system model, the OGY method also requires a learning stage to determine the locations of the saddle-type UFPs using the method of delay-coordinate embedding. There is no guarantee that such learning will require less samples of the process than the method described in this study, or even that its success is ensured for arbitrary nonlinear systems such as those of higher dimension, or those with nonsaddle-type UFPs. Thus the chaos control technique described here can serve as an alternative approach to the OGY method in such situations.

[1] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).

[2] A. Garfinkel, M. L. Spano, W. L. Ditto, and J. N. Weiss, Science **257**, 1230 (1992); S. J. Schiff, K. Jerger, D. H. Duong, T. Chang, M. L. Spano, and W. L. Ditto, Nature (London) **370**, 615 (1994).

[3] J. Singer, Y. Z. Wang, and H. H. Bau, Phys. Rev. Lett. **66**, 1123 (1991).

[4] G. Chen, bibliography available electronically via anonymous ftp to uhoop.egr.uh.edu in the directory /pub/TeX, filename chaos.tex.

[5] D. J. Christini and J. J. Collins, Phys. Rev. E **52**, 5806 (1995).

[6] M. Courtemanche, L. Glass, M. D. Rosengarten, and A. L. Goldberger, Am. J. Physiol. **257**, H693 (1989).

[7] M. E. Brandt and G. Chen, Biol. Cybern. **74**, 1 (1996).

[8] M. E. Brandt and G. Chen, Int. J. Bifurc. Chaos **6**, 715 (1996).

[9] From the viewpoint of chaos control theory, the target trajectory is either a UFP or unstable periodic orbit of the original system. Thus we might refer to control to other fixed points as a form of ''nonstandard control.''

[10] H. K. Khalil, *Nonlinear Systems* (Macmillan, New York, 1992), p. 130.

[11] X. Dong and G. Chen, *Proceedings of the American Control Conference, Chicago, IL,* (IEEE Press, Chicago, 1992), pp. 2234–2235.

[12] Simulation programs were written in *C* and run on a Silicon Graphics Inc. Iris Indigo workstation under IRIX 5.3. For each experiment, the map (1) was iterated 1500 times to reduce any transient system response. Following this loop, an additional 500 iterations were executed with the control enabled. The control gain, g_i was then set to zero (thereby disabling the control) and an additional 500 iterations were made.

[13] Each Gaussian pseudorandom number was computed using the Box-Muller method. See gasdev( ) routine in W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes in C,* 2nd ed. (Cambridge University Press, Cambridge, 1992).

[14] K. Pyragas, Phys. Lett. A **170**, 265 (1992); K. Pyragas and A. Tamaševičius, *ibid.* **180**, 99 (1993).

[15] G. Chen and X. Dong, Int. J. Bifurc. Chaos **2**, 407 (1992).